Problem Review Session 6
PHYS 741

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Disclaimer: The problems below are not my own making but are taken from Pathria's Statistical Mechanics (PSM) and past qualifying exams from UNC (Qual).

Practice Problems

1. **(Qual 2011 SM-5)** A long vertical tube with a cross-section area A contains a mixture of n different ideal gases, each with the same number of particles N, but of different masses m_k, $k = 1, \dots, n$. Find a vertical position of the center of mass of this system in the presence of the Earth's gravity, assuming a constant altitude-independent free fall acceleration g.

2. **(Qual 2012 SM-3)** Consider a classical gas of N identical particles. The energy of the system is given by

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \sum_{i<k} U_{ik}(|\vec{r}_i - \vec{r}_k|).$$

In the dilute (atomic volume $\times N \ll V/N$) and high temperature ($|U| \ll kT$) approximation it can be shown that the partition function can be written as

$$Z(T, V, N) = \frac{1}{N!}\left(\frac{1}{\lambda^3}\right)^N Q_N(V, T) \qquad \text{where} \qquad \lambda = \frac{h}{\sqrt{2\pi m k T}},$$

and the configurational integral $Q_N(V, T)$ is given by

$$Q_N(V, T) = V^N + V^{N-2} \sum_{i<k} \int d^3 r_i \int d^3 r_k (e^{-U_{ik}/kT} - 1).$$

Assume the potential is given by the hard sphere potential

$$U_{ik}(|\vec{r}_i - \vec{r}_k|) = \begin{cases} \infty & |\vec{r}_i - \vec{r}_k| < r_0 \\ 0 & |\vec{r}_i - \vec{r}_k| \geq r_0 \end{cases},$$

where r_0 is the radius of the sphere. Show that the equation of state is given by

$$P\left(V - N\frac{2\pi}{3} r_0^3\right) = NkT.$$

3. **(PSM 4.4)** The probability that a system in the grand canonical ensemble has exactly N particles is given by

$$p(N) = \frac{z^N Q_N(V, T)}{\mathcal{Z}(z, V, T)},$$

where $z = e^{\beta\mu}$ is the fugacity, $Q_N(V,T)$ is the partition function and $\mathcal{Z}(z,V,T)$ is the grand partition function. Verify this statement and show that in the case of a classical, ideal gas the distribution of particles among the members of a grand canonical ensemble is identically a Poisson distribution. Show that

$$\overline{(\Delta N)^2} = kT \left(\frac{\partial \overline{N}}{\partial \mu} \right)_{T,V},$$

where $\overline{N}$ is the average number of particles. Calculate the root mean square fluctuation ΔN for this system from the formula above and from the Poisson distribution, and show that they are the same.

Additional Problem

4. **(PSM 4.7)** Consider a classical system of noninteracting, diatomic molecules enclosed in a box of volume V at temperature T. The Hamiltonian of a single molecule is given by

$$H(\vec{r}_1, \vec{r}_2, \vec{p}_1, \vec{p}_2) = \frac{1}{2m}(p_1^2 + p_2^2) + \frac{K}{2}|\vec{r}_1 - \vec{r}_2|^2,$$

where m and K are constants. Study the thermodynamics of this system, including the dependence of the quantity $\langle |\vec{r}_1 - \vec{r}_2|^2 \rangle$ on T.