# Problem Review Session 4
# PHYS 741

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*Disclaimer:* The problems below are not my own making but are taken from Pathria's Statistical Mechanics (PSM).

## Practice Problems

1. **(PSM 1.8)** Consider a system of quasiparticles whose energy eigenvalues are given by

$$\varepsilon = nh\nu; \qquad n = 0, 1, 2, \ldots .$$

Obtain an asymptotic expression ($N \gg 1$, $E/N \gg 1$) for the number of microstates $\Omega$ of this system for given number $N$ of quasiparticles and a given total energy $E$. Determine the temperature $T$ of the system as a function of $E/N$ and $h\nu$, and examine the situation for which $E/Nh\nu \gg 1$.

2. **(PSM 2.7)**
   (a) Derive an asymptotic expression ($N \gg 1$, $E/N \gg 1$) for the number of ways in which a given energy $E$ can be distributed among a set of $N$ one-dimensional harmonic oscillators, with the energy eigenvalues of the oscillators being $(n + 1/2)\hbar\omega; n = 0, 1, 2, \ldots$.
   (b) Derive the corresponding expression for the "volume" of the relevant region of phase space of this system.
   (c) Establish the correspondence between the two results of (a) and (b), showing that the conversion factor $\omega_0$ is precisely $h^N$. (For those reading Huang's Statistical Mechanics, $\omega_0$ essentially refers to the volume in phase space occupied by a single microstate.)

3. **(PSM 2.8)** Show that

$$V_{3N} = \int \cdots \int_{0 \le \sum_{i=1}^{N} r_i \le R} \prod_{i=1}^{N} \left(4\pi r_i^2 dr_i\right) = \frac{(8\pi R^3)^N}{(3N)!},$$

where $V_{3N}$ is the volume of a $3N$-dimensional hypersphere of radius $R$. Using this results, compute the "volume" of the relevant region of the phase space of an extreme relativistic gas ($\varepsilon = pc$) of $N$ particles moving in three-dimensions. Hence, derive expressions for the various thermodynamic properties of this system (energy, entropy, chemical potential, equation of state, and $\gamma = C_P/C_V$).

*Hint:* Begin with the definition of the $n$-dimensional hypersphere volume

$$V_n = \int \cdots \int_{0 \le \sum_{i=1}^{n} x_i^2 \le R^2} \prod_{i=1}^{n} (dx_i)\,,$$

to find the integral form of $V_{3N}$. Then evaluate the integral by using the fact that $V_{3N} = C_{3N} R^{3N}$, where $C_{3N}$ is a constant of proportionality and use the integral

$$\int_0^{\infty} e^{-r} r^2 dr = 2,$$

to solve for $C_{3N}$.

## Additional Problem

If you want to try another problem similar to PSM 2.8

1. **(PSM 2.9)** Solve the integral

$$\int \cdots \int_{0 \le \sum_{i=1}^{3N} |x_i| \le R} (dx_1 dx_2 \cdots dx_{3N}),$$

and use it to determine the "volume" of the relevant region of the phase space of an extreme relativistic gas ($\varepsilon = pc$) of $3N$ particles moving in one-dimension. Determine, as well, the number of ways of distributing a given energy $E$ among this system of particles and show that (asymptotically) $\omega_0 = h^{3N}$.

## Pathria Notation

Useful Pathria notation

- $\Omega = \Omega(N, E, V)$ refers to the number of microstates that have energy $E$, the number of particles $N$, and occupy a volume $V$.
- $\Gamma = \Gamma(N, E, V; \Delta)$ refers to the number of microstates that have energy $E \le E' \le E + \Delta$, the number of particles $N$, and occupy a volume $V$.
- $\omega$ refers to volume of phase space confined to the region $E \le H(p_i, q_i) \le E + \Delta$. Huang refers to this as $\Gamma(E)$. A useful relation is that $\Gamma = \omega/\omega_0$, where $\omega_0$ is described in the problem above.
- $\Sigma$ refers to the volume of phase space confined to the region $E \le H(p_i, q_i)$, just like in Huang.
- $g(x)$ refers to the density of states of a variable $x$. $x$ can refer to energy, momentum, position, etc. Therefore $g(E)$ in Pathria is essentially the same as $\omega(E)$ in Huang.

# Session 4 Problem 1

PSM 1.8

For this system, the total energy $E$ of $N$ particles is given by

$$\sum_{j=1}^{N} \varepsilon_j = E$$

where $\varepsilon_j = n_j h\nu$ describes the energy of particle $i$

If we "redefine" our energy so that it has an integer value, then we can treat this as a combinatorics problem, where we must find the number of weak compositions that describe how to divide $E^*$ across $N$ particles, where

$$E^* \equiv \sum_{j=1}^{N} n_j = \frac{E}{h\nu}$$

where we see that $E^*$ must be an integer because $n_j$ only takes on integer values $n_j = 0, 1, 2, \ldots$

The number of weak compositions of integer $n$ using $k$ integers is

$$\binom{n+k-1}{n-1} \Rightarrow \Omega = \binom{E^* + N - 1}{N-1} = \frac{(E^* + N - 1)!}{(N-1)!\, E^*!}$$

To find an asymptotic expression, we can take the log and apply Stirling's formula, assuming $E, N \gg 1$ : $\ln n! \simeq n\ln n - n$

$$\Rightarrow \ln\Omega \simeq (E^* + N)\ln(E^* + N) - N\ln N - E^* \ln E^*$$

where I also assume $N-1 \simeq N$ & $E^* + N - 1 \approx E^* + N$

$$\Rightarrow \ln\Omega \simeq E^* \ln(1 + N/E^*) + N \ln(1 + E^*/N)$$

$$\boxed{\Omega = (1+\alpha^{-1})^{\alpha N}(1+\alpha)^N}$$

w/ $\alpha \equiv \dfrac{E}{Nh\nu}$

To determine the temperature dependence, we use the Maxwell relation

$$\frac{1}{T} = \left(\frac{\partial S}{\partial E}\right)_{N,V} = \left(\frac{\partial S}{\partial \alpha}\frac{\partial \alpha}{\partial E}\right)_{N,V} = \frac{1}{Nh\nu}\frac{\partial}{\partial\alpha} k\ln\Omega$$

$$= \frac{k}{h\nu}\ln(1+\alpha^{-1})$$

$$\Rightarrow T = \frac{h\nu}{k\ln\left(1+\frac{Nh\nu}{E}\right)} \simeq \frac{h\nu}{k(Nh\nu/E)} = \boxed{\frac{E}{Nk} = T}$$

Where we assume $\dfrac{E}{Nh\nu} \gg 1$ as stated in the problem $\Rightarrow \ln(1+x) \simeq x$ for $x \ll 1$

Side note: We find that $E = NkT$, which according to the equipartition theorem describes a system w/ just 2 degrees of freedom, like a 1D SHO. This makes sense since 1D SHO has $\varepsilon = (n+\frac{1}{2})h\nu$.

# Session 4 Problem 2

PSM 2.7

a) This problem will begin with a similar procedure to Problem 1

"Redefine" energy to take on integer values

$$E^* \equiv \sum_{j=1}^{N} n_j = \sum_{j=1}^{N} \left( \frac{\varepsilon_j}{\hbar\omega} - \frac{1}{2} \right) = \frac{E}{\hbar\omega} - \frac{N}{2}$$

Again we must count the number of weak compositions that divide $E^*$ among N groups:

where $N \gg 1$

$\therefore N-1 \simeq N$

$$\Omega = \binom{E^* + N - 1}{N-1} \simeq \binom{E^* + N}{N} = \frac{(E^*+N)!}{N!\,E^*!}$$

Taking the log we can use Stirling's approximation: $\ln n! \simeq n\ln n - n$ since $N, E^* \gg 1$

$$\Rightarrow \ln\Omega \simeq (E^*+N)\ln(E^*+N) - N\ln N - E^*\ln E^*$$

This problem asks for an asymptotic express (i.e. when $E/N \gg 1$)

$$\Rightarrow \ln\Omega \simeq + \left(\frac{E}{\hbar\omega} + \frac{N}{2}\right)\left[\ln\frac{E}{\hbar\omega} + \ln\left(1 + \frac{N\hbar\omega}{2E}\right)\right] - N\ln N$$
$$- \left(\frac{E}{\hbar\omega} - \frac{N}{2}\right)\left[\ln\frac{E}{\hbar\omega} + \ln\left(1 - \frac{N\hbar\omega}{2E}\right)\right]$$

$$\Rightarrow \ln\Omega \simeq N\ln E/\hbar\omega - N\ln N + N$$

Using $\ln\left(1 \pm \frac{N\hbar\omega}{2E}\right) \simeq \pm\frac{N\hbar\omega}{2E}$

$$\therefore \boxed{\Omega \simeq \left(\frac{E}{N\hbar\omega}\right)^N}$$

Where we neglect the last N term because $E \gg N$

b) Next we consider the phase space approach, where we look for the shell of phase space enclosed by the energy surfaces defined by $H(q,p) = E$ & $E + \Delta$ where $E$ is the energy of the system & $H$ is the system's Hamiltonian i.e.

$$\tilde{\omega} = \int_{E \le H(q,p) \le E+\Delta} d^N q\, d^N p = \int_{H(q,p) \le E+\Delta} d^N q\, d^N p - \int_{H(q,p) \le E} d^N q\, d^N p$$

note that $\tilde{\omega}$ is called $\Gamma$ in Huang

For this system the Hamiltonian is defined by

$$H = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{m\omega^2 q_i^2}{2} = \sum_{i=1}^{N} \frac{p_i^2}{2m} + \frac{x_i^2}{2m}$$

w/ $x_i \equiv m\omega q_i$

$$\Rightarrow \tilde{\omega} = \left(\frac{1}{m\omega}\right)^N \int_{\sum p_i^2 + x_i^2 \le 2m(E+\Delta)} d^N p \, d^N x \;-\; \left(\frac{1}{m\omega}\right)^N \int_{\sum p_i^2 + x_i^2 \le 2m(E)} d^N p \, d^N x \;=\; \left(\frac{1}{m\omega}\right)^N \left[V_{2N}\left(\sqrt{2m(E+\Delta)}\right) + V_{2N}\left(\sqrt{2mE}\right)\right]$$

where $V_n(R)$ is the volume of an n-dimensional hypersphere with radius R.
The volume is given by

$$V_n(R) = \frac{\pi^{n/2} R^n}{\Gamma(\frac{n}{2}+1)}$$

$$\Rightarrow \tilde{\omega} = \left(\frac{1}{m\omega}\right)^N \frac{[2\pi m(E+\Delta)]^N}{\Gamma(n+1)} - \left(\frac{1}{m\omega}\right)^N \frac{[2\pi E m]^N}{\Gamma(n+1)}$$

Recall $\Gamma(n+1) = n!$

Consider that $E \gg \Delta$ & $E \gg N \Rightarrow (1+\frac{\Delta}{E})^N \simeq 1 + \frac{N\Delta}{E}$

$$\Rightarrow \tilde{\omega} \simeq \frac{1}{N!}\left(\frac{2\pi E}{\omega}\right)^N \left[1 + \frac{N\Delta}{E} - 1\right] = \frac{1}{(N-1)!}\left(\frac{2\pi}{\omega}\right)^N E^{N-1}\Delta$$

Or assuming $N-1 \simeq N$ & applying Stirling's approximation of $n! \simeq n^n e^{-n}$

$$\boxed{\tilde{\omega} \simeq \left(\frac{2\pi E}{N\omega}\right)^N}$$

where we disregard the $\Delta$ term & $e^N$ term, since these will be negligible when we take the log of $\tilde{\omega}$.

c) Our result from part (a) is related to our result from part (b) by

$$\Omega = \tilde{\omega}/\tilde{\omega}_0$$

We can find the normalization $\tilde{\omega}_0$ then by inverting this equation and using our answers from the previous parts

$$\Rightarrow \tilde{\omega}_0 = \tilde{\omega}/\Omega = \left(\frac{\hbar}{2\pi}\right)^N = \boxed{h^N = \tilde{\omega}_0}$$

as expected

# Session 4 Problem 3

PSM 2.8

We can begin with the volume of an n-dimensional hypersphere with radius r, which can be described by the multidimensional integral

$$V_n(R) = \int_{0 \le \sum_{i=1}^{N} x_i^2 \le R^2} d^N x$$

Now if we have a 3N-dimensional space, then every tuple of 3 $x_i$'s can be related to a tuple of spherical coordinates $(r_j, \theta_j, \varphi_j)$ where $j = 1, 2, \dots, N$. Now our volume integral is only restricted for the N $r_j$ coordinates:

$$V_{3N} = \underset{0 \le \sum_{j=1}^{N} r_j \le R}{\int \cdots \int} \prod_{j=1}^{N} r_j^2 dr_j \left( \int_0^{2\pi} d\varphi \right)^N \left( \int_{-1}^{1} d\cos\theta \right)^N$$

$$= \left( 4\pi \int_0^R r^2 dr \right)^N = (4\pi)^N \int_{0 \le \sum r_j \le R} \prod_{j=1}^{N} r_j^2 dr_j \equiv \int dV_{3N}$$

We want to solve this integral by using the known identity of

$$\int_0^\infty e^{-r} r^2 dr = 2$$

$$\Rightarrow 2^N = \left( \int_0^\infty e^{-r} r^2 dr \right)^N = \int_{0 \le \sum_{j=1}^{N} r_j \le \infty} \exp\left[ -\sum_{j=1}^{N} r_j \right] \prod_{j=1}^{N} r_j^2 dr_j ,$$

We see that the product term is related to volume element defined above

$$\prod_{j=1}^{N} r_j^2 dr_j = \frac{dV_{3N}}{(4\pi)^N}$$

We can find an alternate form for $V_{3N}$ by using the ansatz of

$$V_{3N} \sim R^{3N} \Rightarrow V_{3N} = C_{3N} R^{3N}$$

$C_{3N}$ is some constant

$$\Rightarrow dV_{3N} = 3N C_{3N} R^{3N-1}$$

Integral representation of $\Gamma(3N)$

Therefore we can rewrite our integral as

$$2^N = \int_0^\infty e^{-R} \frac{3N C_{3N} R^{3N-1}}{(4\pi)^N} dR = \frac{3N C_{3N}}{(4\pi)^N} \underbrace{\int_0^\infty e^{-R} R^{3N-1} dR}$$

$$= \frac{3N}{(4\pi)^N} C_{3N} \Gamma(3N)$$

$$\Rightarrow C_{3N} = \frac{(8\pi)^N}{(3N)!} \quad \text{or} \quad \boxed{V_{3N} = \frac{(8\pi R^3)^N}{(3N)!}}$$

For an extremely relativistic gas in 3D

$$E = \sum_{i=1}^N p_i c \qquad \text{(where } p_i = |\vec{p}_i| \geq 0\text{)}$$

$\therefore$ we can identify that this problem relates to the derivation in the first half of the problem by taking $R \to E/c$ & $r_i \to p_i$

To find the "volume" of the relevant region of phase space we take the difference of the hypersphere volumes w/ radii $\frac{1}{c}(E+\Delta)$ & $E/c$ respectively.

$$\Rightarrow \tilde{\omega} = \int_{E \leq \sum_{i=1}^N p_i c \leq E+\Delta} d^{3N}p\, d^{3N}q = \left[V_{3N}\left(\frac{E+\Delta}{c}\right) - V_{3N}\left(\frac{E}{c}\right)\right] \underbrace{\int d^{3N}q}_{V^N}$$

$V^N$ for volume $V$

*Note that I am considering these particles as distinguishable, which we will see in later sections gives us the Gibbs paradox*

$$\Rightarrow \tilde{\omega} = \frac{1}{(3N)!}\left(\frac{8\pi V}{c^3}\right)^N \left[(E+\Delta)^{3N} - E^{3N}\right]$$

Recall that we are considering the case where $E \gg \Delta, N$ & $N \gg 1$ $\therefore$ we use $n! \simeq n^n e^{-n}$

$$\Rightarrow \tilde{\omega} = \left(\frac{8\pi V}{27N^3c^3}\right)^N e^{3N} E^{3N} \frac{3N\Delta}{E} \simeq \left(\frac{8\pi V}{27c^3}\right)^N \left(\frac{E}{N}\right)^{3N} e^{3N}$$

when we neglect $\Delta$ b/c $\Delta \ll N, E, V$ & assume $N-1 \simeq N$

But what we want to know are thermodynamic quantities. The entropy is given by

$$S = k \ln \Gamma = k \ln \frac{\tilde{\omega}}{h^{3N}} = \boxed{Nk \ln\left[8\pi V\left(\frac{E}{3Nhc}\right)^3\right] + 3Nk = S}$$

We can then invert our entropy equation to solve for E

$$\Rightarrow \boxed{E = \frac{3Nhc}{(8\pi V)^{1/3}} e^{S/3Nk - 1}}$$

We find temperature using the Maxwell relation

$$T = \left(\frac{\partial E}{\partial S}\right)_{N,V} = \frac{1}{3Nk} E \quad \Rightarrow \quad \boxed{E = 3NkT}$$

The specific heats are given by $C_V = \left(\frac{\partial E}{\partial T}\right)_{N,V}$ & $C_P = T\left(\frac{\partial S}{\partial T}\right)_{N,P}$

$$\Rightarrow \boxed{C_V = 3Nk}$$

But we need to rewrite $S$ in terms of $P$ & $T$ to solve for $C_P$

Pressure is given by the Maxwell relation

$$P = T\left(\frac{\partial S}{\partial V}\right)_{N,E} = \boxed{\frac{NkT}{V} = P}$$

$\therefore$ ideal gas law still holds

$$\Rightarrow S = Nk \ln\left(\frac{8\pi NkT(3NkT)^3}{27PN^3h^3c^3}\right) + 3Nk$$

We see that $S = 4Nk \ln T + f(P,N)$

$\leftarrow$ $f$ is some function that does not depend on $T$

$$\Rightarrow C_P = T\left(\frac{\partial S}{\partial T}\right) = \boxed{4Nk = C_P}$$

Therefore $\boxed{\gamma = \frac{C_P}{C_V} = \frac{4}{3}}$

which is exactly what we expect for a relativistic gas

We can also determine the chemical potential using

$$\mu = -T\left(\frac{\partial S}{\partial N}\right)_{V,E} = \frac{TS}{N} - 3kT$$

$$\Rightarrow \boxed{\mu N = 3NkT - TS = E - TS}$$

as expected