Problem Review Session 3
PHYS 741

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Disclaimer: The problems below are not my own making but are taken from Princeton Problems in Physics (PPP) and past qualifying exams from UNC (Qual).

Past Qualifying Exam Problems

1. **(Qual 2015 SM-2)** Consider a white dwarf star where the number of electrons is N, the mass of the star is $M = 2Nm_p$ (where m_p is the mass of the proton), and the volume of the star is V. The pressure of an ideal Fermi gas is given by

$$P = \frac{8\pi}{3h^3} \int_0^\infty \frac{1}{e^{(\epsilon-\mu)/kT} + 1} \left(p \frac{\partial \epsilon}{\partial p} \right) p^2 dp,$$

   where μ is the chemical potential and ϵ is the relativistic kinetic energy given by

$$\epsilon = m_e c^2 \left\{ \left[1 + \left(\frac{p}{m_e c} \right)^2 \right]^{1/2} - 1 \right\},$$

   where m_e is the mass of the electron and c is the speed of light. It can be shown that the Fermi momentum is given by $p_F = \frac{3N}{8\pi V}^{1/3} h$, where h is the Planck constant. Show that in the $T \to 0$ limit, the radius of the star R is given by the equation

$$\frac{8\pi m_e^4 c^5}{3h^3} \int_0^{\theta_F} \sinh^4 \theta d\theta = \frac{\alpha}{4\pi} \frac{GM^2}{R^4}, \qquad \text{where} \quad m_e c \sinh \theta_F = p_F.$$

   Here $\alpha \simeq 1$ is a known constant, and G is the gravitational constant.

2. **(Qual 2014 SM-1)** Consider a system of N classical distinguishable harmonic oscillators where the Hamiltonian is given by

$$H = \sum_{i=1}^{N} \left(\frac{p_i^2}{2m} + \frac{1}{2} m\omega^2 q_i^2 \right).$$

   (a) Calculate $\Sigma(N, E)$, the total number of microstates with energy less than or equal to E.

   (b) Based on the calculated $\Sigma(N, E)$, show that the entropy is given by

$$S(N, E) = Nk \left[1 + \ln \left(\frac{E}{N\hbar\omega} \right) \right].$$

Practice Problems

3. **(PPP 4.1)** Consider a system of $N \gg 1$ non-interacting particles in which the energy of each particle can assume two and only two distinct values: 0 and E ($E > 0$). Denote by n_0 and n_1 the occupation numbers of the energy levels 0 and E, respectively. The fixed total energy of the system is U.
   (a) Find the entropy of the system
   (b) Find the temperature as a function of U. For what range of values of n_0 is $T < 0$?
   (c) In which direction does heat flow when a system of negative temperature is brought into thermal contact with a system of positive temperature? Why?

4. **(PPP 4.7)** A wire of length l and mass per unit length μ is fixed at both ends and tightened to a tension τ. What is the root mean square fluctuation, in classical statistics, of the midpoint of the wire when it is in equilibrium with a heat bath at temperature T? A useful series is

$$\sum_{m=0}^{\infty} (2m+1)^{-2} = \frac{\pi^2}{8}.$$

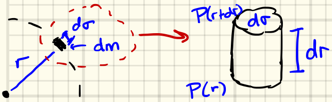
Session 3 Problem 1 Qual 2015 SM-2

The key to this problem is understanding the behavior of the Fermi-Dirac distribution in the $T \to 0$ limit (represents a state of degeneracy) and identifying the relation between pressure, mass, & radius for a stellar structure. I'll begin with the latter because that is a little easier. A star, even a relativistic white dwarf star, can be approximated as a gas/plasma that exists in a self-supported state of hydrodynamic equilibrium, i.e.

$$\frac{dP}{dr} = -\frac{Gm\rho}{r^2}$$

where $m = m(r)$ & is the mass enclosed in a radius r, while $\rho = \rho(r)$ the mass density at radius r.

If you are familiar w/ astrophysics this formula might seem immediately obvious. For the non-astrophysicist, it is a quick relation to derive. In hydrostatic equilibrium, the force due to pressure is balanced by the force of gravity. Imagine a small block of mass dm at a radius r with height dr & cross-section $d\sigma$. From Newton's 2nd law, we simply have



$$\left[P(r+dr) - P(r)\right] d\sigma = -\frac{Gm(r)\,dm}{r^2}$$

recall that gravity also behaves Gauss' flux law & ∴ only mass enclosed contributes to gravity

$$dP\,d\sigma = -\frac{Gm(r)\rho(r)\,dr\,d\sigma}{r^2}$$

$$\rho(r) = \frac{dm}{dV} = \frac{dm}{dr\,d\sigma}$$

So we arrive at $\displaystyle \frac{dP}{dr} = \frac{-Gm(r)\rho(r)}{r^2}$

I will approximate that the density of the white dwarf is about constant

$$\Rightarrow m(r) = \frac{4}{3}\pi r^3 \rho$$

$$\Rightarrow \frac{dP}{dr} = -\frac{4}{3}\pi G \rho^2 r$$

Taking into account that the pressure at the surface of the star is negligible, we integrate from $R \to 0$ to find the central pressure of the star

$$-P_c = -\frac{2}{3}\pi G \rho^2 R^2$$

Using $\rho = \text{const} = M\left(\frac{4}{3}\pi R^3\right)^{-1}$ gives

$$P_c = \frac{3}{8\pi}\frac{GM^2}{R^4} \simeq \frac{\alpha}{4\pi}\frac{GM^2}{R^4}$$

where α is on the same order as $3/2$, but is used to neglect for any mistakes due to our assumptions.

The easier way to derive this is to note that the "self-gravity" of a body goes like GM^2/R^2

$$\Rightarrow \quad P 4\pi R^2 \propto \frac{GM^2}{R^2} \quad \text{or} \quad P \simeq \frac{\alpha}{4\pi}\frac{GM^2}{R^4}$$

from a dimensional analysis point of view.

So the central pressure should be well described by a Fermi gas, w/ the pressure given by the integral in this problem. As $T \to 0$, the Fermi-Dirac distribution will have the following behavior

$$(\epsilon-\mu)/kT \to \infty \qquad \text{for } \epsilon > \mu$$

$$(\epsilon-\mu)/kT \to -\infty \qquad \text{for } \epsilon < \mu$$

$\therefore$ the distribution becomes a step-function that selects all energies less than μ. In the $T \to 0$ case the chemical potential is given by the Fermi energy, since all electrons are lying in all of the lowest energy states (w/out sharing them).

$$\Rightarrow P_c = P = \frac{8\pi}{3h^2}\int_0^{p_F} \frac{\partial \epsilon}{\partial p} p^3 \, dp$$

Using the transformation $p = mc\sinh\theta$, we see that

$$\epsilon = mc^2(\cosh\theta - 1)$$

$$\frac{\partial \epsilon}{\partial p} = \frac{d\theta}{dp}\frac{\partial \epsilon}{\partial \theta} = (mc\cosh\theta)^{-1} mc^2 \sinh\theta$$

Plugging these results into the integral form of P

$$\boxed{P = \frac{8\pi}{3h^2} m^4 c^5 \int_0^{\theta_F} \sinh^4\theta \, d\theta = \frac{\alpha}{4\pi}\frac{GM^2}{R^4}}$$

Session 3 Problem 2 Qual 2014 SM-1

(a) We know that Σ is given by

$$\Sigma(E,N) = \frac{1}{h^N}\int_{H(q,p)\leq E} d^N p\, d^N q$$

For this problem $H(q,p) = \sum_{i=1}^{N}\left(\frac{p_i^2}{2m} + \frac{m\omega^2 q_i^2}{2}\right)$

We see that the quadratic form of the Hamiltonian gives a constraint that looks similar to the constraint that defines the volume of an ellipsoid. But this qual problem only gives you the volume of a n-sphere:

$$V_n(R) = \frac{\pi^{n/2}}{(n/2)!} R^n = \frac{\pi^{n/2}}{(n/2)\Gamma(n/2)} R^n \quad \text{w/ } \Gamma(\nu) = (\nu-1)!$$

Therefore let's recast our Hamiltonian in a more amenable form by defining the new variable $x_i \equiv m\omega q_i$

$$\Rightarrow \Sigma(E,N) = \left(\frac{1}{hm\omega}\right)^N \int_{\sum_{i=1}^{N} p_i^2 + x_i^2 \leq 2mE} d^N p\, d^N q = V_{2N}(R=\sqrt{2mE})$$

$$\boxed{\Sigma(E,N) = \frac{\pi^N}{N!}\left(\frac{2E}{h\omega}\right)^N}$$

b) The entropy is simply related to Σ (in the thermodynamic limit) by

$$S = k \ln \Sigma$$

$$= Nk \ln\pi - k\ln N! + Nk \ln(2E/h\omega)$$

In the thermodynamic limit, $N >> 1$, therefore we can use Stirling's approximation

$$\ln n! = n \ln n - n$$

$$\Rightarrow S = -Nk \ln N + Nk + Nk \ln(2\pi E/h\omega)$$

$$\Rightarrow \boxed{S = Nk\left[1 + \ln\left(\frac{E}{N\hbar\omega}\right)\right]}$$