

## Problem Review Session 2

### PHYS 741

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*Disclaimer:* The problems below are not my own making but are taken from A Guide to Physics Problems: Part 2 (GPP2).

### Practice Problems

- (GPP2 4.11 Liquid-Solid-Liquid)** A small amount of water of mass  $m = 50$  g in a container at temperature  $T = 273$  K is placed inside a vacuum chamber which is evacuated rapidly. As a result, part of the water freezes and becomes ice and the rest becomes vapor.
  - What amount of water initially transforms into ice? The latent heat of fusion (ice to water)  $q_i = 80$  cal/g, and the latent heat of vaporization (water to vapor)  $q_v = 600$  cal/g.
  - A piece of heated metal alloy of mass  $M = 325$  g and original volume  $V = 48$  cm<sup>3</sup> is placed inside the calorimeter together with the ice obtained as a result of the experiment in (a). The density of metal at  $T = 273$  K is  $\rho_0 = 6.8$  g/cm<sup>3</sup>. The thermal capacity is  $C = 0.12$  cal/g K, and the coefficient of linear expansion  $\alpha = 1.1 \times 10^{-5}$  K<sup>-1</sup>. How much ice will have melted when equilibrium is reached?
- (GPP2 4.20 Adiabatic Atmosphere)** The lower 10 – 15 km of the atmosphere, the troposphere, is often in a convective steady state with constant entropy, not constant temperature. ( $PV^\gamma$  is independent of the altitude, where  $\gamma \equiv C_R/C_V$ .)
  - Find the change of temperature in this model with altitude  $dT/dz$ .
  - Estimate  $dT/dz$  in K/km. Consider the average diatomic molecule of air with molar mass  $\mu = 29$  g/mol.
- (GPP2 4.29 Heat Extraction)**
  - A body of mass  $M$  has a temperature-independent specific heat  $C$ . If the body is heated reversibly from a temperature  $T_i$  to a temperature  $T_f$ , what is the change in its entropy?
  - Two such bodies are initially at temperatures of 100 K and 400 K. A reversible engine is used to extract heat with the hotter body as a source and the cooler body as a sink. What is the maximum amount of heat that can be extracted in units of  $MC$ ?
  - The specific heat of water is  $C = 4.2$  J/g K, and its density is 1 g/cm<sup>3</sup>. Calculate the maximum useful work that can be extracted, using as a source 10<sup>3</sup> m<sup>3</sup> of water at 100°C and a lake of temperature 10°C as a sink.
- (GPP2 4.13 Maxwell Boltzmann Averages)**
  - Write the properly normalized Maxwell-Boltzmann distribution  $f(v)$  for finding particles of mass  $m$  with magnitude of velocity in the interval  $[v, v + dv]$  at a temperature  $T$ . (Hint:  $f(E) \sim e^{-E/kT}$ .)
  - What is the most likely speed at temperature  $T$ ?
  - What is the average speed?

- (d) What is the average square speed?
5. **(GPP2 4.35 Poisson Distribution in Ideal Gases)** Consider a monatomic ideal gas of total  $N'$  molecules in a volume  $V'$ . Show that the probability  $P_N$  for the number  $N$  of molecules contained in a small element of  $V$  is given by the Poisson distribution

$$P_N = \frac{e^{-\langle N \rangle} \langle N \rangle^N}{N!}$$

## Another Rocket Problem

1. **(GPP2 4.19 Rocket in Drag)** A rocket has an effective frontal area  $A$  and blasts off with a constant acceleration  $a$  straight up from the surface of the Earth.
- Use either dimensional analysis or an elementary derivation to find out how the atmospheric drag on the rocket should vary as some power(s) of the area  $A$ , the rocket velocity  $v$ , and the atmospheric density  $\rho$  (assuming that we are in the region of high Reynolds numbers).
  - Assume that the atmosphere is isothermal with temperature  $T$ . Derive the variation of the atmospheric density  $\rho$  with height  $z$ . Assume that the gravitational acceleration  $g$  is a constant and that the density at sea level is  $\rho_0$ .
  - Find the height  $h_0$  at which the drag on the rocket is at a maximum.

## Session 2 Problem 1

## GPPZ 4.11 Liquid-Solid-Liquid

a) So let's first take a step back & consider what is happening during this process. Why does the ice evaporate & freeze? When you subject most solutions to a vacuum you would expect them to boil & evaporate, since a vacuum is essentially a  $P \approx 0$  system and most compounds exist in gas phase when  $P \approx 0$  at standard temperature ranges.

However it takes energy or heat to evaporate the liquid. However, because the vacuum chamber was evacuated rapidly, we can approximate that the total energy of the system is conserved. Therefore as the water evaporates, the remaining liquid loses energy. The remaining liquid must either cool or freeze. Under the right conditions we can reach a state where the heat lost to evaporation (vaporization) & freezing (fusion) balance one another and the liquid is completely converted to a solid & vapor.

This is exactly the scenario for this problem, therefore the total latent heat of fusion must equal the total latent heat of vaporization

$$\Rightarrow m_i q_i = m_v q_v$$

We can solve for the mass of the ice by considering that the total mass of the water should be conserved

$$\therefore m_v = m - m_i$$

$$\Rightarrow m_i (q_i + q_v) = m q_v$$
$$m_i = \frac{m q_v}{q_i + q_v} = \frac{m}{1 + q_i/q_v}$$

$$m_i = \frac{750}{17} \text{ g} \approx 44 \text{ g}$$

b) Now we have created a new system where we are worried about the heat exchange between the metal alloy & the ice

$$q_m M = q_i \tilde{m}_i$$

where  $q_m$  is the specific heat gained from the metal & transferred to the ice &  $\tilde{m}_i$  is the amount of ice that has melted when the substances reach equilibrium

But how does  $q_m M$ , i.e. the heat transferred from the metal, affect the metal? When the metal comes in contact w/ the ice, we expect it to quickly cool to the temperature of the ice (273 K). While the metal will not undergo a phase transition like the water, we expect that the volume/density of the material will be altered.

The rate  $C$  which the metal will reach  $T=273\text{ K}$  is related to the thermal capacity  $C$

$$\text{i.e. } C = q_m / \Delta T$$

while the change in volume is related to the coefficient of thermal expansion  $\alpha_V$

$$\text{i.e. } \alpha_V = \frac{1}{V} \left( \frac{\partial V}{\partial T} \right)_P$$

$$\Rightarrow \alpha_V = \left( \frac{\partial \ln V}{\partial T} \right)_P$$

$$\Rightarrow \ln V/V_0 = \alpha_V \Delta T \quad \text{assuming } \alpha_V \approx \text{const.}$$

We want to solve for  $\tilde{m}_i$ ; consequently we must solve for  $q_m$

$$q_m = C \Delta T = \frac{C}{\alpha_V} \ln \frac{V}{V_0} = \frac{C}{\alpha_V} \ln \frac{V \rho_0}{M}$$

However we are not given  $\alpha_V$  in this problem, instead we are given the linear expansion

$$\alpha = \frac{1}{L} \left( \frac{\partial L}{\partial T} \right)_P \quad \text{where } L \text{ is the length along a single direction}$$

Assuming that  $V \approx L^3$ , we can relate  $\alpha_V$  to  $\alpha$

$$\Rightarrow \alpha_V = \frac{1}{L^3} \left( \frac{\partial L^3}{\partial T} \right)_P = \frac{3L^2}{L^3} \left( \frac{\partial L}{\partial T} \right)_P = 3\alpha$$

This leads to the final result of

$$\tilde{m}_i = \frac{MC}{3\alpha q_i} \ln \frac{V \rho_0}{M} \approx 63 \text{ g}$$

Side note: Other solutions use the relation

$$\frac{V}{V_0} = 1 + \alpha_V \Delta T \quad \leftarrow \text{this comes from assuming } |\alpha_V T| \ll 1 \text{ \& expanding the exp as a Taylor series. Gives same answer.}$$

## Session 2 Problem 2

GPP2 4.20

## Adiabatic Atmosphere

- a) To make things easier we will begin by assuming that the troposphere is governed by the ideal gas equation of state

$$PV = nRT \Rightarrow T = \frac{PV}{nR}$$

This should be a reasonable assumption since the atmosphere is relatively dilute. But what do we actually mean by the volume  $V$  & number of moles  $n$ ? It might be more illustrative to think that for  $n$  moles of gas, they occupy a pressure  $P$ , volume  $V$ , & temperature  $T$  @ a certain altitude  $z$ .

$\therefore$  for a chosen value  $n = n'$

$$P = P(z, n') \quad T = T(z, n') \quad V = V(z, n')$$

$$\Rightarrow \frac{dT}{dz} = \frac{1}{n'R} \frac{d}{dz}(PV) = \frac{1}{n'R} \left[ V \frac{dP}{dz} + P \frac{dV}{dz} \right]$$

Hydrostatic equilibrium (since the atmosphere is in a steady state) gives us

$$\frac{dP}{dz} = -\rho g$$

where  $\rho$  is the density of our  $n'$  moles of atmosphere @ altitude  $z$ , such that

$$\rho = \frac{n'\mu}{V}$$

where we assume the atmospheric molecules can be described by a mean molecular weight  $\mu$ .

To solve for  $dV/dz$ , we use our knowledge that the troposphere experiences convective heating & therefore our  $n'$  moles of atmosphere behave adiabatically

$$\Rightarrow PV^\gamma = \text{const} \quad \text{or} \quad P^{1/\gamma} V = \text{const} \equiv \alpha$$

$$\Rightarrow P \frac{dV}{dz} = -\frac{\alpha}{\gamma} P^{1/\gamma} \frac{dP}{dz} = -\frac{V}{\gamma} \frac{dP}{dz}$$

$$\therefore \frac{dT}{dz} = \frac{V}{n'R} \left( \frac{\gamma-1}{\gamma} \right) \frac{dP}{dz} = \left( \frac{\gamma-1}{\gamma} \right) \frac{\mu}{R} g = \frac{dT}{dz}$$

b) Now it's just a matter of determining  $\gamma$ . Recall that the equipartition theorem tells us that the internal energy of an ideal gas is given by

$$U = \frac{1}{2} n R f T$$

where  $f$  is the degrees of freedom for the system. We also know that

$$C_V = \left( \frac{\partial U}{\partial T} \right)_V \quad \& \quad C_P - C_V = nR \quad \text{for an ideal gas}$$

$$\Rightarrow \gamma = C_P / C_V = 1 + nR / C_V = 1 + 2/f = \frac{f+2}{f}$$

A diatomic molecule has 5 degrees of freedom: 2 rotational & 3 translational

$$\Rightarrow \gamma = 7/5$$

$$\text{So } \frac{dT}{dz} = \left( \frac{1 - 7/5}{7/5} \right) \frac{29 \text{ g/mol} \cdot 9.8 \text{ m/s}^2}{8.3 \times 10^3 \text{ g m}^2 \text{ s}^{-2} \text{ K}^{-1} \text{ mol}^{-1}} \times \frac{1000 \text{ m}}{\text{km}} = \boxed{-9.8 \text{ K} = \frac{dT}{dz}}$$

This is only  
off by a factor of 2<sup>0</sup>  $\uparrow$   
from the actual temperature gradient of the  
troposphere which is remarkable for such a simple  
model

## Session 2 Problem 3

## GPP2 4.29 Heat Extraction

a) As we have discussed in previous problems

$$MC = \frac{dQ}{dT} \quad \text{since } C \text{ is temperature independent}$$

Because this is a reversible process, we also have the equality

$$dQ = T dS$$

$$\Rightarrow dS = MC \frac{dT}{T}$$

Integrating both sides we find  $\Delta S = MC \ln \frac{T_f}{T_i}$

b) The heat extracted by the system is given by the difference of initial & final internal energies of the reservoirs, which is given by the heat

$$\begin{aligned} Q_{\text{ext}} &= Q_f - Q_i \\ &= (Q_H + Q_C) - (Q_{H,f} + Q_{C,f}) \end{aligned}$$

when  $Q_H$  = heat of hot reservoir,  $Q_C$  = heat of cold reservoir &  $Q_{i,f}$  represents heat in the final state of the engine. Defining  $T_{H,f}$  &  $T_{C,f}$  as the final temperatures of the hot & cold reservoirs respectively, we have

$$\begin{aligned} Q_{\text{ext}} &= MC(T_H + T_C) - MC(T_{H,f} + T_{C,f}) \\ &= [T_H + T_C - T_H e^{\Delta S_H/MC} - T_C e^{\Delta S_C/MC}] MC \end{aligned}$$

where we have used our result from part (a) & represent the change in entropy of the hot & cold reservoirs by  $\Delta S_H$  &  $\Delta S_C$  respectively. Now, we know that

$$\Delta S = \Delta S_H + \Delta S_C \geq 0$$

so let's rewrite  $\Delta S_H = \Delta S - \Delta S_C$

$$\Rightarrow [T_H + T_C - T_H e^{\Delta S/MC} - T_H e^{-\Delta S_C/MC} - T_C e^{\Delta S_C/MC}] MC$$

$\therefore$  We see that  $Q_{\text{ext}}$  is maximized when  $\Delta S = 0$ , otherwise the  $T_H e^{\Delta S/MC}$  term grows

$$\Rightarrow \Delta S_H = MC \ln \left( \frac{T_{H,f}}{T_H} \right) = -\Delta S_C = MC \ln \left( \frac{T_C}{T_{C,f}} \right)$$

$$\Rightarrow T_H T_C = T_{H,f} T_{C,f}$$

So our expression for the extracted heat becomes

$$Q_{\text{ext}} = (T_H + T_C - \frac{T_H T_C}{T_{H,f}} - T_{H,f}) MC$$

Now I have not assumed a value for  $T_{H,f}$  or  $T_{C,f}$ , but we expect that  $T_{H,f} = T_{C,f}$  minimizes our expression, which we can show by minimizing the equation above

$$\frac{dQ}{dT_{H,f}} = 0 = MC \left( \frac{T_H T_C}{T_{H,f}^2} - 1 \right)$$

$$\Rightarrow T_{H,f}^2 = T_H T_C \quad \text{or} \quad T_{H,f} = \sqrt{T_H T_C}$$

$$\Rightarrow T_{C,f} = T_{H,f} = \sqrt{T_H T_C}$$

$\therefore$  we finally have that

$$Q_{\text{ext}} = (T_H + T_C - 2\sqrt{T_H T_C}) MC = \boxed{100 MC \cdot K}$$

↳ We want to maximize work, therefore we want to maximize efficiency and a Carnot engine operating @  $T_H$  &  $T_C$  will achieve this exact result:

$$\eta = \frac{W}{Q_{\text{abs}}} = 1 - \frac{T_C}{T_H}$$

$$\Rightarrow W = \left(1 - \frac{T_C}{T_H}\right) Q_{\text{abs}}$$

But... our realistic system cannot actually operate as a true Carnot engine. Why? While the cold reservoir is large and its temperature can be considered roughly constant, the same is not true for our hot reservoir. A Carnot engine begins w/ an isothermal expansion at temperature  $T_H$ . This absorbs heat from the hot reservoir, therefore causing the reservoir temperature to slightly drop. As a result, once the engine goes through a full cycle, the hot reservoir is no longer @ temperature  $T_H$  due to

$$dQ_{\text{abs}} = -dQ_H = -MC dT$$

where  $-dQ_H$  is the heat lost by the hot reservoir &  $dT$  is its temperature change.

So let's reimagine this as a series of Carnot engines w/ each engine in the series having a new hot reservoir temperature. While the lake also gains heat & therefore its temperature will also increase, however this temperature change  $\delta T = \delta Q / M_L C$  where  $M_L C$  is the mass of the lake &  $\delta Q$  is the heat absorbed by the lake. Since  $M_L C \gg \delta Q$ , we will neglect the temperature change of the lake @ each step in the cycle & will only account for it at the end.

Therefore the work done by each cycle with  $T_H = T$

$$\Rightarrow dW_{CT} = \eta(T) dQ_{abs} = -\left(1 - \frac{T_c}{T}\right) MCdT$$

Or integrating all the way down to  $T_H = T_c + \Delta$ , where this is the final temperature of the two reservoirs.

$$\begin{aligned}\Rightarrow W_{tot} &= \int_{T_c + \Delta}^{T_H} \left(1 - \frac{T_c}{T}\right) dT \\ &= MC \left[ T_H - (T_c + \Delta) - T_c \ln\left(\frac{T_c + \Delta}{T_H}\right) \right] \\ &= MC \left[ T_H - T_c - T_c \ln\left(\frac{T_H}{T_c}\right) \right] \quad \text{using } \ln(1+x) \approx x\end{aligned}$$

$$W = 2.8 \times 10^{11} \text{ J}$$

The alternative approach is to follow the process of part (b) by identifying that maximum work is extracted when  $\Delta S = 0$  so initial & final entropies must be the same.

$$\begin{aligned}\Rightarrow S &= MC \ln T_H + M_L C \ln T_c = (M + M_L) C \ln(T_c + \Delta) \\ &\approx (M + M_L) C \left[ \ln T_c + \frac{\Delta}{T_c} \right]\end{aligned}$$

$$\Rightarrow \Delta = \frac{M}{M + M_L} T_c \ln \frac{T_H}{T_c}$$

The total work can then be calculated by comparing the initial & final energies since  $\Delta Q = 0$  from  $\Delta S = 0$ .

$$\Rightarrow W = U_i - U_f = MCT_H + M_LCT_c - (M + M_L)C(T_c + \Delta)$$

$$W_{tot} = MC(T_H - T_c - T_c \ln \frac{T_H}{T_c})$$

# Session 2 Problem 4

## GPPZ 4.13 Maxwell Boltzmann Averages

(a)

As we've seen before, Maxwell-Boltzmann distributions have the form

$$f \sim e^{-E/kT}$$

In this case, we will rewrite  $E = \frac{1}{2}mv^2$

$$\Rightarrow \tilde{f}(v) = C e^{-mv^2/2kT}$$

$$\text{where } 1 = \int d^3v \tilde{f}(v)$$

Therefore we just need to normalize the distribution solve for C

$$\begin{aligned} \Rightarrow 1 &= C \int_{-\infty}^{\infty} d^3v e^{-mv^2/2kT} \\ &= 4\pi C \int_0^{\infty} dv v^2 e^{-mv^2/2kT} \end{aligned}$$

This is a good chance to explain how one solves this integral. Define

$$\begin{aligned} I &= \int_{-\infty}^{\infty} dx e^{-ax^2} \\ \Rightarrow I^2 &= \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} dx dy e^{-a(x^2+y^2)} = \int_0^{\infty} \int_0^{2\pi} dr d\theta r e^{-ar^2} \\ &= 2\pi \int_0^{\infty} dr r e^{-ar^2} = \frac{\pi}{a} \int_0^{\infty} du e^{-u} \quad \begin{array}{l} u = ar^2 \\ du = 2ar dr \end{array} \\ &= \frac{\pi}{a} \\ \Rightarrow I &= \sqrt{\frac{\pi}{a}}, \quad \frac{dI}{da} = \sqrt{\frac{\pi}{4a^3}} \quad \& \quad \int_0^{\infty} e^{-au^2} du = \frac{1}{2} I \end{aligned}$$

So we have

$$1 = 4\pi C \sqrt{\pi}/4 \left(\frac{2kT}{m}\right)^{3/2} \quad \text{or} \quad C = \left(\frac{m}{2\pi kT}\right)^{3/2}$$

$$\Rightarrow f(v) = 4\pi v^2 \left(\frac{m}{2\pi kT}\right)^{3/2} e^{-\frac{mv^2}{2kT}}$$

$$\text{where } \int_0^{\infty} dv f(v) = 1$$

(b) The most likely speed is given by the maximum of  $f(v)$ . So first we find the zeros by taking the derivative

$$\Rightarrow \frac{df}{dv} = 2 \frac{f(v)}{v} - \frac{2vm}{2kT} f(v) = 0$$

$$\Rightarrow v^2 = \frac{2kT}{m} \quad \therefore \quad v = \sqrt{\frac{2kT}{m}}$$

(c) The average velocity is given by

$$\langle v \rangle = \int_0^{\infty} dv v f(v) = 4\pi C \int_0^{\infty} dv v^3 e^{-mv^2/2kT}$$

Setting  $u = mv^2/2kT$  so  $du = mvdv/kT$  or  $v^2 = 2kT/m$

$$\Rightarrow \langle v \rangle = 2 \left( \frac{kT}{m} \right)^2 4\pi C \int_0^{\infty} du u e^{-au}$$

where  $a=1$ , but we introduce it for convenience

$$= -8\pi \left( \frac{kT}{m} \right)^2 C \frac{d}{da} \left[ \int_0^{\infty} du e^{-au} \right]$$

$$= -8\pi \left( \frac{kT}{m} \right)^2 C \frac{d}{da} \left( \frac{1}{a} \right) = \frac{8\pi C}{a^2} \left( \frac{kT}{m} \right)^2$$

But  $a=1$  &  $C = (m/2\pi kT)^{3/2}$

$$\Rightarrow \langle v \rangle = 8\pi C \left( \frac{kT}{m} \right)^2 \quad \text{or} \quad \langle v \rangle = \sqrt{\frac{8kT}{\pi m}}$$

(d) For a quick result, one can recall the equipartition function, for which

$$\langle E \rangle = \frac{3}{2} kT \Rightarrow \langle \frac{1}{2} mv^2 \rangle = \frac{3}{2} kT$$

$$\Rightarrow \langle v^2 \rangle = \frac{3kT}{m}$$

Alternatively, one must evaluate the integral  $4\pi C \int dv v^2 f(v)$

## Session 2 Problem 5

## GPPZ 4.35 Poisson Distribution in Ideal Gases

The probability of finding 1-particle in a volume  $V$  out of an ideal gas of  $N'$  particles in volume  $V'$  is given by  $\omega_1 = V/V'$ .

Therefore the probability of not finding a particle in  $V$   $\bar{\omega}_1 = 1 - \omega_1$ . This means that the probability of finding  $N$  molecules in volume  $V$  &  $N' - N$  molecules not in  $V$

$$\Rightarrow P = \frac{N'!}{N!(N'-N)!} \omega_1^N \bar{\omega}_1^{N'-N} = \frac{N'!}{N!(N'-N)!} \left(\frac{V}{V'}\right)^N \left(1 - \frac{V}{V'}\right)^{N'-N}$$

When the prefactor takes into account all of the ways we can arrange the  $N'$  particles between the  $N$  particles in  $V$  &  $N' - N$  particles in the rest of  $V'$ .

A useful identity is that, for  $N' \gg N$

$$\frac{N'!}{(N'-N)!} = N'(N'-1)(N'-2)\dots(N'-N+1) \approx N'^N \quad \text{since each term is} \\ \approx N'$$

$$\Rightarrow P \approx \frac{N'^N}{N!} \left(\frac{V}{V'}\right)^N \left(1 - \frac{V}{V'}\right)^{N'} = \frac{\langle N \rangle^N}{N!} \left(1 - \frac{\langle N \rangle}{N'}\right)^{N'}$$

by recognizing that  $\langle N \rangle = N'V/V'$ . Additionally

$$N' \ln\left(1 - \frac{\langle N \rangle}{N'}\right) \approx -\langle N \rangle$$

So we can approximate the second factor by  $e^{-\langle N \rangle}$

$$\Rightarrow P \approx \frac{\langle N \rangle^N}{N!} e^{-\langle N \rangle}$$