

# Problem Review Session 1

## PHYS 741

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*Disclaimer:* The problems below are not my own making but are taken from A Guide to Physics Problems: Part 2 (GPP2) and Princeton Problems in Physics (PPP).

### Practice Problems

1. **(GPP2 4.12 Hydrogen Rocket)** The reaction chamber of a rocket engine is supplied with a mass flow rate  $m$  of hydrogen and sufficient oxygen to allow complete burning of the fuel. The cross section of the chamber is  $A$ , and the pressure at the cross section is  $P$  with temperature  $T$ . Calculate the force that the chamber is able to provide.
2. **(PPP 4.4 Phase Coexistence)** The temperature of a long vertical column of a particular substance is  $T$  everywhere. Below a certain height  $h(T)$  the substance is solid, whereas above  $h(T)$  it is in a liquid phase. Calculate the density difference  $\Delta\rho = \rho_s - \rho_l$  between the solid and liquid ( $|\Delta\rho| \ll \rho_s$ ) in terms of  $L$  (the latent heat of fusion per unit mass),  $dh/dT$ ,  $T$ ,  $\rho_l$  and  $g$ , the acceleration due to gravity.

3. **(GPP2 4.51 Nonideal Gas Equation)** A gas obeys the equation of state

$$P = Nk \left( \frac{T}{V} + \frac{B(T)}{V^2} \right)$$

where  $B(T)$  is a function of the temperature  $T$  only. The gas is initially at temperature  $T$  and volume  $V_0$  and is expanded isothermally and reversibly to volume  $V_1 = 2V_0$ .

- (a) Find the work done in the expansion.
  - (b) Find the heat absorbed in the expansion.
4. **(PPP 4.5 Otto Cycle)** The operation of a gasoline engine is (roughly) similar to the Otto cycle (Fig 1), where:
    - $A \rightarrow B$ : Gas compressed adiabatically
    - $B \rightarrow C$ : Gas heated isochorically (constant volume; corresponds to combustion of gasoline)
    - $C \rightarrow D$ : Gas expanded adiabatically (power stroke)
    - $D \rightarrow A$ : Gas cooled isochorically

Compute the efficiency of the Otto cycle for an ideal gas (with temperature-independent heat capacities) as a function of the compression ratio  $V_A/V_B$ , and the heat capacity per particle  $C_V$ .

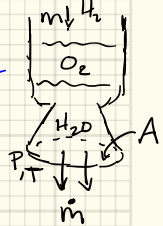
5. **(GPP2 4.32 Joule Cycle)** Find the efficiency of the Joule cycle, consisting of two adiabats and two isobars (see Fig 2). Assume that the heat capacities of the gas  $C_P$  and  $C_V$  are constant.
6. **(GPP2 4.53 Critical Parameters)** Consider a system described by the Dietrici equation of state

$$P(V - nB) = nN_A k T e^{-nA/N_A k T V}$$

where  $A$ ,  $B$ ,  $k$ ,  $N_A$  and  $R$  are constants and  $P$ ,  $V$ ,  $T$ , and  $n$  are the pressure, volume, temperature, and number of moles. Calculate the critical parameters, i.e., the values of  $P$ ,  $V$ , and  $T$  at the critical point.

# Session 1 Problem 1 GPP2 4.12

## Hydrogen Rocket



We begin by considering the forces that act on the chamber

$$F_{\text{tot}} = F_{\text{out}} + F_p$$

where  $F_{\text{out}}$  refers to the force produced by the mass outflow due to fuel combustion, while  $F_p$  is the force due to the pressure differential at the interface of the chamber & the external environment. For a traditional rocket, where the combustion products exit through a nozzle to maximize the outflow velocity, the second term is negligible. For this problem

$$F_p \approx PA$$

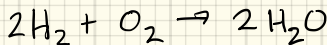
but we will neglect it & focus on the reaction thrust from the mass flow from the chamber,  $F_{\text{out}}$

As one may recall from classical mechanics

$$\vec{F} = \frac{d\vec{p}}{dt} = \dot{m}\vec{v} + m\vec{a}$$

In this case  $\vec{F}_{\text{out}}$  actually arises from the former term rather than the latter

- ∴ we need to calculate  $\vec{v}$  the velocity of material leaving the chamber and  $\dot{m}$ , the mass outflow rate. To determine the outflow rate we can first consider the material actually being produced in the combustion chamber. The reaction in the chamber would be



- ∴ the amount of hydrogen that enters the system is equal to the amount of water vapor that leaves the chamber, i.e.

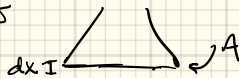
$$\dot{n}_{\text{H}_2} = \dot{n}_{\text{H}_2\text{O}}$$

$$\text{Now } m = \dot{n}_{\text{H}_2} M_{\text{H}_2} = (2g) \dot{n}_{\text{H}_2} = (2g) \dot{n}_{\text{H}_2\text{O}}$$

$$\text{while } \dot{m} = \dot{n}_{\text{H}_2\text{O}} M_{\text{H}_2\text{O}} = (18g) \dot{n}_{\text{H}_2\text{O}} = 9m$$

So now we know  $\dot{m}$  & just need to solve for  $\vec{v}$ . Imagine that at a given moment, the  $\text{H}_2\text{O}$  produced in the chamber takes up a volume  $V$ . Then the volume decreases by  $dV$  given by

$$dV = A dx$$



where  $A$  is the cross-section of the chamber &  $x$  is the distance along the the direction of mass flow. Therefore we can express the mass outflow rate as

$$\dot{m} = \frac{d}{dt}(\rho V) = \rho \frac{dV}{dt} = \rho A \frac{dx}{dt} = \rho A v$$

where  $\rho$  is the density of  $\text{H}_2\text{O}$  & we recognize  $\frac{dx}{dt} = v$

$$\Rightarrow v = \frac{\dot{m}}{\rho A}$$

So now we must solve for  $\rho$

$$\Rightarrow \rho = \frac{n_{\text{H}_2\text{O}} M_{\text{H}_2\text{O}}}{V} = \frac{P M_{\text{H}_2\text{O}}}{RT}$$

where we assumed  $\text{H}_2\text{O}$  product behaves like an ideal gas w/ equation of state

$$\frac{P}{RT} = \frac{n}{V}$$

$$\Rightarrow v = \frac{\dot{m} RT}{P M_{\text{H}_2\text{O}} A}$$

$$\therefore F \approx \frac{8 \text{ km}^2 RT}{M_{\text{H}_2\text{O}} PA}$$

Notice that this is maximized by reducing  $PA$ , which is related to  $F_p$ . This is why rocket nozzles are designed to reduce  $PA$  & as a result, why  $F_p$  is negligible compared to  $F_{out}$ .

## Session 1 Problem 2

We begin with the Clausius-Clapeyron equation, since this defines the phase transition or phase coexistence curve on a  $P$ - $T$  diagram & we have a substance existing in two distinct phases in this problem.

$$\frac{dP}{dT} = \frac{L}{T\Delta v}$$

Recall that  $v$  is the volume per a unit mass (i.e. inverse density),  $\Delta v$  describes the specific volume change of the phase transition, and  $L = T\Delta s$  is the latent heat w/  $s$  as the entropy per unit mass &  $\Delta s$  is the specific entropy change from the phase transition.

For this problem we are given  $L =$  latent heat of fusion which means we are concerned about the transition from a solid to liquid.

$$\Rightarrow \Delta s = s_L - s_S \quad \& \quad \Delta v = v_L - v_S = \frac{1}{\rho_L} - \frac{1}{\rho_S}$$

Rearranging the second relation, we see that

$$\Delta v = \frac{\rho_S - \rho_L}{\rho_S \rho_L} = \frac{\Delta \rho}{\rho_S \rho_L}$$

$$\therefore \Delta p = \frac{L}{T \frac{dP}{dT}} \rho_S \rho_L$$

Expanding  $dP/dT$  in terms of the height function  $h(T)$

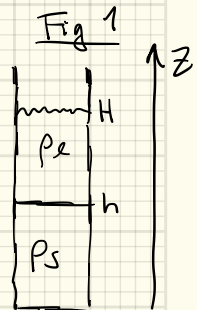
$$\Rightarrow \frac{dP}{dT} = \frac{dh}{dT} \frac{dP}{dh}$$

Therefore we need to determine  $dP/dh$ . I'll demonstrate a longer derivation first. We know that the pressure @ the surface of a liquid in a tube is merely

$$P = \rho_L g \Delta z \quad \text{where } \Delta z \text{ is the height of the liquid}$$

From Fig 1, we see  $\Delta z = H - h$

$$\Rightarrow \frac{dP}{dh} = \rho_L g \left( \frac{dH}{dh} - 1 \right) \quad \text{where } H \text{ also depends on } h \text{ by non-trivially depending on } T, \text{ just like } h$$



To solve for  $dH/dh$  we can consider that, while the height of the mixture is not constant as a function of temperature, the total mass of the system is constant, i.e.

$$m = \rho_s A h + \rho_e A (H - h) \quad \text{where } A \text{ is the cross-section of the tube}$$

$$\Rightarrow \frac{dm}{dh} = 0 = \rho_s A + \rho_e A \left( \frac{dH}{dh} - 1 \right)$$

$$\therefore \frac{dP}{dh} = -\rho_s g$$

which gives us the final result

$$\Delta p = \frac{-\rho_e L}{g T (dh/dT)}$$

# Session 1 Problem 3

## GPPZ 4.57 Nonideal Gas Equation

We have the equation of state  $P = Nk \left( \frac{T}{V} + \frac{B(T)}{V^2} \right)$

a)

Work done in the expansion:  $W = - \int_{V_i}^{V_f} P dV$  w/  $V_i = V_0$  &  $V_f = 2V_0$  for this problem

Note we pick up a negative sign because we are concerned about the work done in the system, not by the system on its environment

$$\Rightarrow W = - \int_{V_0}^{2V_0} Nk \left( \frac{T}{V} + \frac{B(T)}{V^2} \right) dV$$

$$W = -NkT \ln 2 - \frac{NkB(T)}{2V_0}$$

b) So now we are looking for the heat absorbed, which is given by

$$Q_{in} = \int_{V_0}^{2V_0} T dS$$

Since we are dealing w/ a reversible process. This is an isothermal expansion so  $T = \text{const}$  & independent of  $S$

$$\Rightarrow Q_{in} = T [S(2V_0) - S(V_0)]$$

To relate  $S$  &  $V$  we must first relate  $S$  &  $P$ . We can do this by considering the Helmholtz free energy

$$dA = -SdT - PdV$$

which gives us the Maxwell relation

$$\left( \frac{\partial S}{\partial V} \right)_T = \left( \frac{\partial P}{\partial T} \right)_V$$

$$\Rightarrow S = Nk \int \left( \frac{1}{V} + \frac{B'(T)}{V^2} \right) dV = Nk \ln \left( \frac{V}{V_c} \right) - \frac{NkB'(T)}{V} + C(T)$$

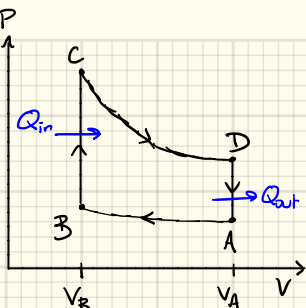
where  $V_c$  is a constant of integration as well as  $C(T)$  due to holding  $T$  const and  $B'(T) = \frac{\partial B}{\partial T}$

Evaluating  $S(V)$  for the initial & final volumes, we see that

$$Q_{in} = NkT \left( \ln 2 + \frac{B'(T)}{2V_0} \right)$$

# Session 1 Problem 4

Otto Cycle:



We want to compute the efficiency, which is defined by

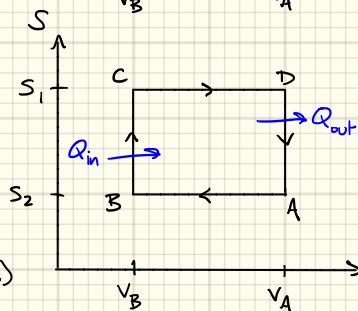
$$\eta = \frac{W_{\text{tot}}}{Q_{\text{in}}}$$

We are dealing w/ a cyclic process, so  $dU=0$

$$\Rightarrow dW = dQ$$

The process is also reversible

$$\Rightarrow dQ = TdS$$



From the diagrams, we see that heat is absorbed/rejected in the transformations from  $B \rightarrow C$  &  $D \rightarrow A$  respectively (since  $C \rightarrow D$  &  $A \rightarrow B$  are adiabatic)

$$\therefore W_{\text{tot}} = Q_{B \rightarrow C} + Q_{D \rightarrow A}$$

$$Q_{\text{in}} = Q_{B \rightarrow C}$$

$$\Rightarrow \eta = 1 + \frac{Q_{D \rightarrow A}}{Q_{B \rightarrow C}}$$

\* Note in the diagrams we use  $Q_{\text{out}} = -Q_{D \rightarrow A} > 0$

To calculate the heat for these two transformations, we must determine  $T$  as a function of  $S$ . So we begin with the first law

$$dU = TdS - PdV = TdS - \frac{NkT}{V}dV \quad \text{assuming an ideal gas}$$

$$\text{Because } C_V \text{ is temperature independent } C_V = \left(\frac{dU}{dT}\right)_V \Rightarrow dU = C_V dT$$

$$\Rightarrow dS = C_V \frac{dT}{T} + Nk \frac{dV}{V}$$

Integrating this equation we find that

$$S = C_V \ln T + Nk \ln V + \alpha$$

where  $\alpha$  is an integration constant

Inverting to get  $S$  as a function of  $T$

$$\Rightarrow T = \beta e^{\frac{S/C_V}{V^{-Nk/C_V}}}$$

where  $\beta = e^{-\alpha/C_V}$



Now we can insert our temperature function into the integrals over entropy

$$\Rightarrow Q_{B \rightarrow C} = \int_{S_2}^{S_1} T(S, V_B) dS = \beta V_B^{-Nk/c_V} \int_{S_2}^{S_1} e^{S/c_V} dS$$

$$= \beta V_B^{-Nk/c_V} C_V (e^{S_1/c_V} - e^{S_2/c_V})$$

$$Q_{D \rightarrow A} = \int_{S_1}^{S_2} T(S, V_A) dS = \beta V_A^{-Nk/c_V} C_V (e^{S_2/c_V} - e^{S_1/c_V})$$

Consequently, plugging into our expression for the efficiency we find that

$$\eta = 1 + \frac{Q_{D \rightarrow A}}{Q_{B \rightarrow C}}$$

$$\eta = 1 - \left( \frac{V_A}{V_B} \right)^{-Nk/c_V}$$

As a useful exercise you can also calculate the efficiency from

$$\eta = \frac{W_{\text{tot}}}{Q_{\text{in}}}$$

by considering  $W_{\text{tot}} = W_{A \rightarrow B} + W_{C \rightarrow D}$  and calculating

$$W_{i \rightarrow j} = \int_{V_i}^{V_j} P dV = Nk \int_{V_i}^{V_j} \frac{T(V, S_{i,j})}{V} dV \text{ for each transformation}$$

This should give you the same value of  $\eta$  as above.

# Session 1 Problem 5

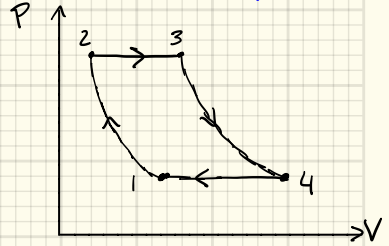
## GPP2 4.32 Joule Cycle

Recall that the efficiency is given by

$$\eta = \frac{W_{\text{tot}}}{Q_{\text{in}}}$$

where  $Q_{\text{in}}$  is the heat absorbed &  $W_{\text{tot}}$  is the total work done by the system

From the diagram, we see that  $Q_{\text{in}} = Q_{2 \rightarrow 3}$   
where  $Q_{2 \rightarrow 3}$  is the heat absorbed from  $2 \rightarrow 3$ .



This is a cyclic process  $\therefore dU=0 \Rightarrow dQ=dW$

$$\Rightarrow W_{\text{tot}} = Q_{2 \rightarrow 3} + Q_{4 \rightarrow 1} \quad \text{since } 3 \rightarrow 4 \text{ \& } 1 \rightarrow 2 \text{ are adiabatic}$$

Because these are constant pressure processes &  $C_p$  is temperature independent we can take advantage of the relation

$$C_p = \left( \frac{\partial Q}{\partial T} \right)_P$$

by rewriting it as  $dQ = C_p dT \Rightarrow Q_{i \rightarrow j} = C_p(T_j - T_i)$

where  $T_i$  &  $T_j$  are the temperatures at points  $i$  &  $j$  on the PV plot where  $i, j = 1, 2, 3, 4$ .

$$\therefore \eta = \frac{W_{\text{tot}}}{Q_{\text{in}}} = 1 + \frac{Q_{4 \rightarrow 1}}{Q_{2 \rightarrow 3}} = 1 + \frac{T_1 - T_4}{T_3 - T_2}$$

But we want our answer in terms of  $P_1$  &  $P_2$   $\therefore$  we first use the ideal gas law

$$\Rightarrow \eta = 1 + \frac{P_1}{P_2} \left( \frac{V_1 - V_4}{V_3 - V_2} \right) \quad \text{when we used the fact that } P_1 = P_4 \text{ \& } P_2 = P_3$$

We can eliminate  $V$  by considering the adiabatic transformations, along which

$$PV^\gamma = \text{const} \quad \text{where } \gamma \equiv C_p/C_v$$

$$\Rightarrow V \sim P^{-1/\gamma} \Rightarrow \frac{V_4}{V_3} = \left( \frac{P_2}{P_1} \right)^{1/\gamma} \quad \frac{V_1}{V_2} = \left( \frac{P_2}{P_1} \right)^{1/\gamma}$$

$$\therefore \text{we find that } \eta = 1 + \frac{P_1}{P_2} \left[ \frac{(V_2 - V_3)(P_2/P_1)^{1/\gamma}}{V_3 - V_2} \right] \quad \text{or } \boxed{\eta = 1 - \left( \frac{P_1}{P_2} \right)^{(\gamma-1)/\gamma}}$$

# Session 1 Problem 6 GPPZ 4.53 Critical Parameters

Recall that a critical point meets the conditions

$$\left(\frac{\partial P}{\partial V}\right)_{T=T_c} = 0 \quad \& \quad \left(\frac{\partial^2 P}{\partial V^2}\right)_{T=T_c} = 0$$

$$\begin{aligned} \therefore P &= nN_A kT e^{-nA/N_A kTV} (V-nB)^{-1} \\ &= \alpha(T) e^{-\beta(T)/V} (V-nB)^{-1} \quad \text{with } \alpha(T) \equiv nN_A kT \\ &\quad \beta(T) \equiv nA/N_A kT \end{aligned}$$

$$\Rightarrow \left(\frac{\partial P}{\partial V}\right)_{T_c} = -\alpha e^{-\beta/V_c} (V_c-nB)^{-2} + \alpha e^{-\beta/V_c} \beta_c V_c^{-2} (V_c-nB)^{-1} = 0$$

$$\begin{aligned} \text{which gives us } \beta_c V_c^{-2} (V_c-nB)^{-1} &= (V_c-nB)^{-2} \quad \text{with } \alpha_c \equiv \alpha(T_c) \\ \Rightarrow \beta_c^{-1} &= (V_c-nB) V_c^{-2} \quad \beta_c \equiv \beta(T_c) \end{aligned}$$

Likewise we have

$$\begin{aligned} \left(\frac{\partial^2 P}{\partial V^2}\right)_{T_c} &= 2\alpha e^{-\beta/V_c} (V_c-nB)^{-2} - \alpha e^{-\beta/V_c} \beta_c V_c^{-2} (V_c-nB)^{-2} - \alpha e^{-\beta/V_c} \beta_c (V_c-nB)^{-2} V_c^{-2} \\ &\quad - 2\alpha e^{-\beta/V_c} \beta_c V_c^{-3} (V_c-nB)^{-1} + \alpha e^{-\beta/V_c} \beta_c^2 V_c^{-4} (V_c-nB)^{-1} = 0 \end{aligned}$$

Multiply by  $V_c (V_c-nB)^3$  & recall that  $\beta_c = V_c^2 (V_c-nB)^{-1}$

$$\Rightarrow 2V_c^2 - 2V_c - 2(V_c-nB) + V_c = 0$$

$$\boxed{V_c = 2nB}$$

$$\Rightarrow \beta_c^{-1} = N_A kT_c / nA = nB / (4n^2 B^2) = (4nB)^{-1}$$

$$\Rightarrow \boxed{T_c = \frac{A}{4N_A kB}}$$

Plugging into our equation for pressure we find

$$P_c = nN_A \left(\frac{A}{4n kB}\right) e^{-\frac{nA/nA}{A} \left(\frac{4n kB}{A}\right) \left(\frac{1}{2nB}\right)} (nB)^{-1} = \boxed{\frac{A}{4B^2} e^{-2} = P_c}$$