HW #4 Solutions

Problem 6.1 (unofficially)

6.1) In class you were asked to show that in the thermodynamic limit $S = k\ln\Gamma(E) \simeq k\ln\Sigma(E) \simeq k\ln\omega(E)$, meaning they only differ on the level of $\ln N$ or smaller.

Recall that ① $\Gamma(E) \equiv \int_{E \le H(q,p) \le E+\Delta} d^{\mathcal{N}}q\, d^{\mathcal{N}}p$, ② $\Sigma(E) \equiv \int_{H(q,p)\le E} d^{\mathcal{N}}q\, d^{\mathcal{N}}p$, ③ $\omega(E) = \left.\frac{\partial \Sigma}{\partial E}\right|_E$

where $\mathcal{N} = d\cdot N$, where N is the number of particles & d is the number of dimensions accessible to those particles. From these definitions $\Sigma(E)$ & $\omega(E)$ can be related to $\Gamma(E)$ by

$$\Gamma(E) = \Sigma(E+\Delta) - \Sigma(E) = \omega(E)\Delta$$

While there are multiples way to demonstrate that $\frac{S}{k}$ can be calculated by taking the log of any of these 3 quantities, I find it easiest to first consider the behavior of $\Sigma(E)$ according to Eqn ②

Eqn ② essentially defines $\Sigma(E)$ as the volume in phase space that is enclosed by the hypersurface defined by $H(q_i, p_i) = E$. Therefore the energy defines a scale "length" for the volume that spans $2\mathcal{N}$ dimensions

Therefore we expect $\Sigma(E) \sim E^{\alpha\mathcal{N}}$ when $\alpha > 0$ & $\mathcal{O}(1)$ and depends on the form of the Hamiltonian.

Following from this relation

$$\begin{aligned}\Gamma(E) &= \Sigma(E+\Delta) - \Sigma(E) \sim (E+\Delta)^{\alpha\mathcal{N}} - E^{\alpha\mathcal{N}} \\ &\sim E^{\alpha\mathcal{N}}\left(1+\frac{\Delta}{E}\right)^{\alpha\mathcal{N}} - E^{\alpha\mathcal{N}} \\ &\sim E^{\alpha\mathcal{N}-1}\Delta\alpha\mathcal{N} \quad \text{by taking into account } \frac{\Delta}{E} \ll 1\end{aligned}$$

Taking the $\ln$ of $\Gamma(E)$ & $\Sigma(E)$ & $N\to\infty$, so $\mathcal{N}\to\infty$ & $\alpha\mathcal{N}-1 \simeq \alpha\mathcal{N}$

$$\Rightarrow \ln\Gamma(E) \sim \alpha\mathcal{N}\ln E + \ln(\alpha\Delta) + \ln\mathcal{N}$$

$$\ln\Sigma(E) \sim \alpha\mathcal{N}\ln E$$

← negligible b/c $\alpha, \Delta \ll N, E$

Therefore $\boxed{\ln\Gamma(E) - \ln\Sigma(E) \sim \mathcal{O}(\ln N)}$

To demonstrate the thermodynamic equivalence of these statements with $\ln \omega(E)$ consider

$$\Gamma(E) = \omega(E)\Delta$$

$$\Rightarrow \ln \Gamma(E) = \ln \omega(E) + \ln \Delta$$

negligible

Therefore we see that as $N \to \infty$

$$\boxed{\ln \Sigma(E) \simeq \ln \Gamma(E) \simeq \ln \omega(E)}$$