HW #3 Solutions

Problems:
4.2, 4.4, 4.5, 4.6

4.2

4.2 A cylindrical column of gas of given temperature rotates about a fixed axis with constant angular velocity. Find the equilibrium distribution function.

Huang asserts that for a dilute gas experiencing a conservative force given by

$$\vec{F} = -\vec{\nabla}\phi(r) \qquad \text{(Huang 4.26)}$$

Then the equilibrium distribution function is given by

$$f(\vec{p},\vec{r}) = f_0(\vec{p})\, e^{-\phi(r)/kT} \qquad \text{(Huang 4.27)}$$

Another way to think about this is just that (for most cases)

$$f(\vec{p},\vec{q}) \sim e^{-H/kT}$$

where H is the Hamiltonian of the system. This system, in its frame of reference, experiences a centripetal force described by the potential

$$\phi = \frac{1}{2} m\omega^2 r^2$$

$n_0 \equiv$ number density @ $r=0$

$$\boxed{\Rightarrow f(p,r) = n_0 (2\pi m k T)^{-3/2}\, e^{-p^2/2mkT - m^2\omega^2 r^2/2mkT}}$$

4.4

For this problem I set $c=1$

4.4 Using relativistic dynamics for gas molecules find, for a dilute gas of zero total momentum,

(*a*) the equilibrium distribution function;

(*b*) the equation of state.

Answer. *PV* is independent of the volume. Hence it is *NkT* by definition of *T*.

Essentially this problem is asking up to follow Maxwell's heuristic derivation of the Maxwell distribution, but this time for relativistic molecules. Therefore, like in the book, we begin by recognizing that our equilibrium distribution $f(\vec{p})$, must satisfy the Boltzmann transport equation

$$\Rightarrow \log f(\vec{p}_1) + \log f(\vec{p}_2) = \log f(\vec{p}_1{}') + \log f(\vec{p}_2{}') \quad \text{(Huang 4.12)}$$

As noted by Huang, this takes the form of a conservation equation. Therefore each distribution can be expressed as a function of conserved quantities. Energy and momentum are conserved for the system we are considered, but since the bulk momentum of the gas is zero, our distribution should just depend on energy and other constants that we will need to relate to other thermodynamic quantities. For relativistic particles, we consider their four-momentum p^μ, which in the laboratory frame, takes the form

$$p^\mu = (E, p_x, p_y, p_z)$$

Taking advantage of $p^\mu p_\mu = -m \Rightarrow E^2 = p^2 + m^2$

$$\therefore \log f(\vec{p}) = -\beta E(p) + \alpha$$

$$\Rightarrow f(\vec{p}) = Ce^{-\beta E(p)} = Ce^{-\beta\sqrt{p^2+m^2}}$$

when α & β are constants that I assume are >0. Why? If $\beta<0$ then this distribution would favor ∞ energies, which seems unrealistic & we expect the distribution to be positive $\therefore C>0$.

So now we want to solve for C & β by relating this distribution to physical quantities.

Recall that $f(\vec{p})$ was defined such that $f(\vec{p})d^3q\,d^3p$ gives us the number of molecules in the volume $d^3q\,d^3p$ of the phase space. Note that other authors define the distribution function $\tilde{f}$ as the likelihood that a particle exists in the volume $d^3q\,d^3p \Rightarrow f(\vec{p}) = N\tilde{f}(\vec{p})$ where N is the number of molecules in the gas.

$$\therefore N = \int d^3q\,d^3p\, f(p) \quad \text{or} \quad \frac{N}{V} \equiv n = \int d^3p\, f(p)$$

since f does not depend on position.

$$\Rightarrow n = C\int_{-\infty}^{\infty}dp_x\int_{-\infty}^{\infty}dp_y\int_{-\infty}^{\infty}dp_z\, e^{-\beta(p^2+m^2)^{1/2}} = C\int_0^{\pi}dp_\theta\int_0^{2\pi}dp_\phi\int_0^{\infty}dp\cos\theta\, p^2 e^{-\beta(p^2+m^2)^{1/2}}$$

w/ $p_r^2 = p_x^2 + p_y^2 + p_z^2 = p^2$, $p_\theta = \arccos\frac{p_z}{p}$, $p_\phi = \arctan\frac{p_y}{p_x}$

when we transformed to a spherical coordinate projection of the momentum on the second integral.

$$\Rightarrow \quad n = 4\pi C \int_0^\infty p^2 e^{-\beta(p^2+m^2)^{1/2}} dp$$

We can simplify the argument of the exponential by integrating over energy instead of momentum;

$$E^2 = p^2 + m^2 \Rightarrow 2E\,dE = 2p\,dp$$

(1) $$\Rightarrow n = 4\pi C \int_m^\infty p\,E\,e^{-\beta E} dE = 4\pi C \int_m^\infty E(E^2-m^2)^{1/2} e^{-\beta E} dE$$

This integral looks something along the lines of a Bessel function:

(2) $$K_\nu(z) = \frac{\pi^{1/2}\left(\frac{z}{2}\right)^\nu}{\Gamma(\nu+1/2)} \int_1^\infty e^{-zt}(t^2-1)^{\nu-1/2} dt$$ Modified Bessel function of the 2nd kind

But you can also just plug this into Mathematica and it will return

(3) $$n = 4\pi C\left[\frac{m^2 K_2(\beta m)}{\beta}\right]$$

If you did not use Mathematica, then you could compute this from the integral form of the Bessel function in Eqn (2). The recursion relations of the modified Bessel functions also help:

$$K'_\nu(z) = K_{\nu+1}(z) + \frac{\nu}{z}K_\nu(z)$$

$$K'_\nu(z) = K_{\nu-1}(z) - \frac{\nu}{z}K_\nu(z)$$

From this we can immediately identify that

$$\boxed{C = \frac{n\beta}{4\pi m^2 K_2(\beta m)}}$$

So now we just need to determine β. Now I know that this problem asks to find the equation of state of the gas, which is a bit confusing, because the β-term, in the non-relativistic case, is identified by taking

(4) $$P = nkT$$

as an experimental fact. Therefore the equation of state is the same in the relativistic case, when we define the temperature so that Eqn. (4) is correct.

What I think this problem really wants is for you to identify an equation of state that is a function of P, n, & ϵ, the average energy, in place of T, the temperature. So let's solve for the average energy

$$(5) \quad \epsilon = \frac{\int d^3p\, E\, f(p)}{\int d^3p\, f(p)} = \frac{4\pi C}{n} \int_m^\infty E^2 (E^2 - m^2)^{1/2} e^{-\beta E}\, dE$$

which can also be expressed as

$$\epsilon = -\frac{1}{n}\frac{\partial n}{\partial \beta} = -\frac{\partial \ln n}{\partial \beta}$$

You can either plug this derivative into Mathematica or use the Bessel function recursion relations to then find

$$(6) \quad \boxed{\epsilon = \frac{3}{\beta} + \frac{m K_1(m\beta)}{K_2(m\beta)}}$$

Finally, let's determine a relation for P, the pressure. Huang relates the pressure to the momentum distribution by

$$P = \int d^3p\, p_x v_x f(\vec{p}) \qquad \text{(Huang 4.20)}$$

For a relativistic particle $p^i = \gamma m v^i$ & $E = \gamma m$ where $\gamma = (1-v^2)^{-1/2}$

$$\Rightarrow P = \int d^3p \frac{p_x^2}{E} f(\vec{p}) = \frac{1}{3}\int d^3p \frac{p^2}{E} f(\vec{p})$$

Again, switching integration variables so that we are integrating over energy, we find

$$P = \frac{4\pi C}{3}\int_m^\infty dE\, (E^2-m^2)^{3/2} e^{-\beta E}$$

$$= \left(\frac{4\pi C}{3}\int_m^\infty dE\, E^2 (E^2-m^2)^{1/2} e^{-\beta E}\right) - \left(\frac{4\pi C}{3}\int_m^\infty dE\, m^2 (E^2-m^2)^{1/2} e^{-\beta E}\right)$$

$$= \frac{n\epsilon}{3} - \frac{4m^3\pi C}{3\beta} K_1(\beta m)$$

where we solved the first integral by relating it to Eqn (5) and the second integral by using Eqn. (2)

Plugging in our value of C, we find that

$$\boxed{P = \frac{n\epsilon}{3} - \frac{mn}{3}\frac{K_1(\beta m)}{K_2(\beta m)}}$$

Substituting Eqn (6) for ϵ, we find

$$P = n\beta^{-1} \Rightarrow \boxed{\beta^{-1} = kT}$$

If we want a distribution that obeys the ideal gas law. Therefore our distribution is given by

$$\boxed{f(p) = \frac{n}{4\pi m^2 kT K_2(m/kT)} e^{-\sqrt{p^2+m^2}/kT}}$$

which is typically referred to as the Maxwell-Jüttner distribution

Acknowledging to relativistic ideal gases exist @ high temperatures, we can take the limit of $T \to \infty$ or $\beta \to 0$ & simplify our expressions for energy & pressure by noting that

$$K_\nu(z) \sim \frac{1}{2}\Gamma(\nu)\left(\frac{1}{2}z\right)^{-\nu} \quad \text{as } z \to 0$$

$$\Rightarrow \frac{K_1(\beta m)}{K_2(\beta m)} \simeq \frac{\beta m}{2} + \mathcal{O}(\beta^3) \quad \text{as } \beta \to 0$$

$$\boxed{\begin{aligned} &\Rightarrow \epsilon \simeq 3kT + \frac{m^2}{2kT} \simeq 3kT \\ &\Rightarrow P = nkT \simeq \frac{1}{3}n\epsilon \end{aligned}}$$

@ leading order which is what we measure for hot, relativistic, dilute gases

4.5

4.5 (*a*) *Estimate* the probability that a stamp (mass = 0.1 g) resting on a desk top at room temperature (300 K) will spontaneously fly up to a height of 10^{-8} cm above the desk top.

Hint. Think not of one stamp but of an infinite number of noninteracting stamps placed side by side. Formulate an argument showing that these stamps obey the Maxwell-Boltzmann distribution.

Answer. Let m = mass of stamps, h = height, g = acceleration of gravity. Probability $\approx e^{-mgh/kT}$

If we imagine that a stamp is just one of an infinite number of stamp particles that make-up some stamp gas, then we can describe the distribution of the system using the Maxwell-Boltzmann distribution in terms of the energy

$$f(E) = C E^{1/2} e^{-E/kT}$$

C = normalization constant

The probability of a particle being in a state w/ energy E' is given by

* I know this is flawed so see my notes at the end for comments on this approach *

$$P(E') = \frac{(E')^{1/2} e^{-E'/kT}}{\int_0^\infty E^{1/2} e^{-E/kT}\, dE}$$

Therefore the probability of a stamp being at a height h is given by

$$P(mgh) = \frac{(mgh)^{1/2} e^{-mgh/kT}}{\int_0^\infty (mgz)^{1/2} e^{-mgz}\, mg\,dz} = \sqrt{\frac{2mgh}{\pi k^3 T^3}}\; e^{-mgh/kT}$$

Considering all of the values given in the problem & the constants:

$m = 0.1$ g, $k = 1.38\times10^{-16}$ cm^2 g s^{-2} K^{-1}

$g = 980$ cm/s^2, $T = 300$ K

$h = 10^{-8}$ cm

Because of the value of k, the exponential will dominate, so it is easier to look at the log of the probability

$$\ln P = -\frac{mgh}{kT} + \frac{1}{2}\ln\left(\frac{2mgh}{\pi k^3 T^3}\right)$$

this term ~40 which is still << 10^7

I also address why this piece is not dimensionless on the next page

$$\simeq -\frac{mgh}{kT} = \boxed{-2.4\times10^{7} = \ln P}$$

Side note to 4.5

When dealing w/ distributions, the probability is well defined over an interval. For instance, we should ask what is the probability that stamp has some energy between E & E+Δ:

$$P(E \leq \tilde{E} \leq E+\Delta) = \frac{\int_E^{E+\Delta} dE'\, E'^{1/2} e^{-E'/kT}}{\int_0^\infty dE'\, E'^{1/2} e^{E'/kT}} = \frac{\partial P}{\partial E}\Delta \quad \text{for } \Delta \ll E$$

If you integrate the numerator & expand in powers of Δ/E, you find that

$$P(E \leq \tilde{E} \leq E+\Delta) \simeq \frac{E'^{1/2} e^{-E'/kT} \Delta}{\int_0^\infty dE'\, E'^{1/2} e^{-E'/kT}} \quad \text{or} \quad \frac{\partial P}{\partial E} = E'^{1/2} e^{-E'/kT}$$

Therefore the probability we calculate in Problem 4.5 is a probability density with respect to the energy. This neglect for detail is not a cause for too much concern however since

$$\ln P = \ln \tilde{P} + \ln \Delta$$

when $\tilde{P}$ is the probability density (i.e. the equation I use to solve 4.5). The second term will be extremely small & so the probability is effectively given by the probability density.

Also note that our probability function is a bit ridiculous because when $z \to 0$, $E \to 0$ &

$$\tilde{P}(E=0) = 0$$

Which essentially says that the probability of the stamp lying on the table is 0 based on the way we approached the problem. No need to worry, however. If you calculate the stamp's most probable position:

$$\langle z \rangle = \frac{\int z\, E\, e^{-E/kT} dE}{\int E\, e^{-E/kT} dE} = \frac{3}{2}\frac{kT}{mg} \simeq \boxed{6 \times 10^{-18}\ \text{m}}$$

Since this height is on a length scale smaller than a proton, we can consider $\langle z \rangle \simeq 0$.

4.6

4.6 A room of volume $3 \times 3 \times 3$ m^3 is under standard conditions (atmospheric pressure and 300 K).

(*a*) *Estimate* the probability that at any instant of time a 1-cm^3 volume anywhere within this room becomes totally devoid of air because of spontaneous statistical fluctuations.

(*b*) estimate the same for a 1-Å^3 volume.

Answer. Let N = total number of air molecules, V = volume of room, v = the volume devoid of air. Probability $\approx e^{-N(v/V)}$

(a)

If we consider the Maxwell-Boltzmann distribution for the gas:

$$f(\vec{p}) = \frac{n}{(2\pi mkT)^{3/2}} e^{-p^2/2mkT}$$

We see that it does not depend on position. Therefore the likelihood that a particle of the gas is in a volume $\tilde{v}$ in a room of volume V, is expressed simply by $\tilde{v}/V$. Therefore the probability that the particle or N particles are <u>NOT</u> in the volume $\tilde{v}$ is

$$P = (1 - \tilde{v}/V)^N$$

where P is probability. I will use $\tilde{P}$ for pressure

Note $\tilde{P} = 1\text{ atm} = 1\times10^5\ N/m^2$, $k = 1.38\times10^{-23}\ N\cdot m/K$, $\tilde{v} = 1\times10^{-6}\ m^3$, $V = 27\ m^3$

$$\Rightarrow \tilde{v}/V \simeq 4\times10^{-8} \Rightarrow P = (1-\tilde{v}/V)^N \ll 1$$

Therefore we can take the log of the probability to get an idea of how small it is

$$\Rightarrow \ln P = N \ln(1 - \tilde{v}/V)$$

$$\simeq -N\tilde{v}/V$$

Since $\tilde{v}/V \ll 1$ & $\ln(1-x) = -x + \mathcal{O}(x^2)$

Assuming that this is an ideal gas

$$\Rightarrow \boxed{\ln P = -\frac{\tilde{P}}{kT}\tilde{v} \simeq -2.4\times10^{19}}$$

Obviously this seems very unlikely

$$P \sim 10^{-10^{19}}$$

(b)

Now we just adjust our value for $\tilde{v} = (1\ Å)^3$. An angstrom is 10^{-10} m

$$\Rightarrow \tilde{v} = (10^{-10}\ m)^3 = 10^{-30}\ m$$

$$\Rightarrow \ln P = -2.4\times10^{-5}$$

$$\Rightarrow \boxed{P \simeq 1 - 2.4\times10^{-5}}$$

Again noting that $10^{-5} \ll 1$ & $e^x \simeq 1 + x + \mathcal{O}(x^2)$

There it is very likely that this smaller volume will not be inhabited by any air molecules.